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Age Composition and Other Demographic Measures of Immigrants and Their Descendants

1. Introduction

BELOW replacement fertility is no longer rare as it is prevalent today m a large number of countries. In terms of population size, the U.S. with a population of *244* million, tops the list of such countries followed by Japan with *122* million, West Germany with 61 million, Italy, the U *K,* and France each with 56 million. In all, there are *29* countries, characterized by such low levels of fertility (Population Reference Bureau, 1987) that account for more than one sixth of the total world population of five billion. The entire North American continent, 80 percent of the people of Oceania and 75 percent of the population of Europe, excluding the U.S.S.R , are currently experiencing such a fertility pattern, the continuation of which, while other things remain constant, will, in the long run, result in a declining trend in population size. Twenty of these countries happen to *be* located in Europe and together, they account for the largest number of 380 million in one continent followed by 270 million in the U.S. and Canada in North America. It is no wonder that several European demographers voiced their concern about the future of population in their countries at the 1987 European Population Conference hosted by the Government of Finland.

Understandably, the nature of the problem assumes different proportions that are primarily determined by the magnitude of the net reproduction rate and the immigration-emigration policy of the country concerned. The U.S., Canada, Australia and the U.K. are among countries that admit qualified immigrants on a regular basis, while most of the other countries are virtually closed to immigration.

During the past few years several studies (Basavarajappa, Verma and Pictor, 1984; Espenshade *et al.,* 1982; Mitra and Cerone, 1986; Mitra, 1983; Sivamurthy, 1982) were undertaken to look into the effect of immigration on population growth. The simple example in which a constant number of immigrants with a specific age composition enter every year in a country experiencing an unchanging schedule of age-specific fertility and mortality rates was found amenable to simplification by the mathematics of stable population. In particular, when the condition of below replacement fertility prevails, it was found that the population eventually becomes stationary while its size is determined by the size and the age composition of the immigrants as also by the net reproduction rate. This, of course, depends on the assumption that the immigrants and/or their descendants experience the same vital rates as that of the host country.

As time moves forward, the population composition of such a country will continue to exhibit increasing proportions of immigrants and their descendants of several generations. Since the latter may be alternatively labelled as the native population (by virtue of their country of birth) a comparison of the demographic characteristics of the former to the latter merits careful attention. This is so because one of the determinants of the impact of the immigrants' cultural backgrounds on that of the host population is the relative size of the former to the latter, the relevance of which for formulating national policies on immigration hardly needs any emphasis. What follows next is the derivation of the limiting form of the age distributions and other demographic measures of the immigrants under certain simplifying assumptions. Unless otherwise stated, our illustrations will be based only on one sex, namely the female.

2. The Age Distribution of Immigrants

Let us begin by writing $I(x, t)$ for the number immigrating at time t who are x years old. Suppose

$$I(x, t) > 0 \quad \text{for } A < x \leqslant B \tag{1}$$

$$\quad = 0 \text{ otherwise}$$

where

$$0 \leqslant A \leqslant B < \alpha \tag{2}$$

a being the span of life. Thus the model assumes that no one immigrates at an age which is less than A or greater than B. Denoting the time of start of the process by $t = 0$, the *agi* distribution of the immigrant population at any subsequent time f can be expressed as

$$P[I(x, t)] = 0 \quad \text{for } 0 \leqslant x < A \tag{3}$$

$$= \int_0^t \frac{I(x-h,\, t-h)\, l(x)}{l(x-h)}\, dh \tag{4}$$

for $A \leqslant x \leqslant B$ and $t \leqslant x - A$ assuming that mortality rates remain constant over time. For $t > x - A$ (4) can be written as

$$P[I(x, t)] = \int_0^{x-A} \frac{I(x-h,\, t-h)\, l(x)}{l(x-h)}\, dh \tag{5}$$

If in addition, the number immigrating at age x is also independent of time (5) can be expressed simply as

$$P[I(x)] = \int_0^{x-A} \frac{I(x-h)\, l(x)}{l(x-h)}\, dh, \quad A < x < B \tag{6}$$

For a complete description of the age composition of the immigrants we need the expression for $x > B$ which assumes the limiting value also for $t \geqslant x - A$ as

$$P[I(x)] = \int_{x-B}^{x-A} \frac{I(x-h)\, l(x)}{l(x-h)}\, dh, \quad x > B \tag{7}$$

It is easy to see from (6) that the cumulative nature of those expressions will manifest in an age distribution that is unimodal rather than monotonically declining.

2 (a). *Particular case* : $I(x) = kl(x)$ *for* $A < x \leqslant B$ *and* $= 0$ *Otherwise*

Simpler forms of the generalized expressions given in the preceding section can be obtained when the number *I(x)* immigrating at age x is proportional to the stationary population *l(x)* that also describes their mortality experiences. In that case we can write $l(x) = kl(x)$ where k is the constant of proportionality. Substitution of this pattern in (3), (6) and (7) gives for $t > x - A$.

$$P[I(x)] = 0 \quad \text{for } 0 < x < A \tag{8}$$

$$= kl(x) \int_0^{x-A} dh = k(x - A)\, l(x), \quad A \leqslant x < B \tag{9}$$

$$= k(B - A)\, l(x) \quad \text{for } x > B \tag{10}$$

Adding (9) and (10) we obtain the limiting size of the immigrant population for $t > \alpha - A$ as

$$P(I) = k \left[\int_A^B (x - A)\, l(x)\, dx + (B - A) \int_B^\alpha l(x)\, dx \right]$$

$$= k \left[\int_A^B x l(x)\, dx - A \int_A^\alpha l(x)\, dx + B \int_B^\alpha l(x)\, dx \right] \tag{11}$$

Using the conventional life table notation

$$T(y) = \int_y^\alpha (x)\, dx \tag{12}$$

such that

$$dT(y) = -\, l(y)\, dy \tag{13}$$

we can express the total number immigrating at any time as

$$I = k \int_A^B l(x)\, dx = k[T(A) - T(B)] \tag{14}$$

and (11), the limiting size of the immigrant population as

$$P(I) = k\left[- \int_A^B x dT(x) - AT(A) + BT(B) \right] \tag{15}$$

which after integration by parts simplifies further into

$$P(I) = k \int_A^B T(x)\, dx \tag{16}$$

For the special case when $A = 0$ and $B = \alpha$, that is to say, when immigrants enter at all ages (14) can be reduced further as

$$I = kT(0) = ke(0) \tag{17}$$

and $$P[I(x)] = kxl(x) \tag{18}$$

where $T(0) = e(0)$ is the expectation of life at birth. Similarly, the limiting size of the immigrant population namely, $P(I)$ in (16) can be simplified as

$$P(I) = k \int_0^{\alpha} T(X)\, dx$$

which after integration by parts or from (18) can be a alternatively expressed as

$$P(I) = k \int_0^{\alpha} xl(x)\, dx = ke(0)\,\bar{x} \tag{19}$$

where x is the average age of the stationary population. The unimodal nature of this age distribution becomes readily apparent by differentiating $xl(x)$ and equating the result to zero. The reciprocal of the modal age turns out to be equal to the force of mortality at that age.

2 (b). *Particular Case* : $I(x) = ke^{-cx}l(x)$ for $A \leqslant x \leqslant B$ *and* $= 0$ *otherwise*

It may be mentioned that although the model presented in 2(a) can be derived from the current model by putting c = 0, it is being treated separately for the special nature of some of its important characteristics. Following procedures similar to those in the preceding section, the counterparts of (8), (9) and (10) can be written as

$$P_e[I(x)] = 0 \quad \text{for } 0 \leqslant x < A \tag{20}$$

$$= \frac{k}{c}[e^{-ca} - e^{cx}]\, l(x) \quad \text{for } A \leqslant x \leqslant B \tag{21}$$

$$= \frac{k}{c}[e^{-cA} - e^{-cB}]\, l(x) \quad \text{for } x > B \tag{22}$$

Further, the number immigrating at any time can be written as

$$I_e = k \int_A^B e^{-cx} l(x)\, dx \tag{23}$$

The limiting size of the total immigrant population can be obtained by adding (18) and (19) and simplifying as

$$P_e(I) = k \int_A^B e^{-cx} T(x)\, dx \tag{24}$$

In the special case when $A = 0$ and $B = \alpha$, (23) and (24) can be simplified somewhat by noting the similarity between the stable population model and the integral in (23). In terms of the stable model, the integral over the entire limit is equivalent to the reciprocal of the intrinsic birth rate when c is looked upon as the intrinsic rate of growth. In that case, writing

$$\int_0^{\alpha} e^{-cx} l(x)\, dx = \frac{1}{m} \qquad (25)$$

we can express I_e as

$$I_e = \frac{k}{m} \qquad (26)$$

and $P_e(I)$ after integration by parts as

$$P_e(I) = \frac{1}{c}\left[e(o) - \frac{1}{m} \right] \qquad (27)$$

2 (c). *Particular Case : $I(x) = k$ for $A \leqslant x \leqslant B$ and $= 0$ Otherwise*

This is an interesting example in which the number of immigrants does not vary by age in a selected age interval. Thus the limiting size of the age distribution of the immigrant population will be given by

$$P[I(x)] = 0 \quad \text{for } 0 \leqslant x < A \qquad (28)$$

$$= kl(x) \int_A^x \frac{dy}{l(y)} \quad \text{for } A \leqslant x \leqslant B \qquad (29)$$

$$= kl(x) \int_A^B \frac{dy}{l(y)} \quad \text{for } x > B \qquad (30)$$

Therefore, the limiting size of the total immigrant population can be obtained as the sum of (28), (29) and (30), i.e.,

$$P(I) = k \int_A^B l(x) \int_A^x \frac{dy}{l(y)}\, dx + k \int_A^B \frac{dy}{l(y)} \int_B^{\alpha} l(x)\, dx \qquad (31)$$

$$\text{Let } f(x) = \int_A^x \frac{dy}{l(y)} \qquad (32)$$

such that

$$f'(x) = \frac{1}{l(x)} \tag{33}$$

Therefore, (31) can be rewritten as

$$P(I) = k \int_A^B - f(x)\, dx(x) + kf(B)\, T(B)$$

which after integration by parts simplified into

$$P(I) = k\, [-f(x)\, T(x)]_A^B + k \int_A^B \frac{T(x)}{l(x)}\, dx + kf(B)\, T(B) = k \int_A^B e(x)\, dx \tag{34}$$

Correspondingly, the number immigrating at any instant of time is given by

$$I = k\,(B - A) \tag{35}$$

The modification of (34) and (35) when $A = 0$ and $B = \alpha$ can be seen through appropriate substitutions in those expressions.

3. Deaths among Immigrants

A general expression for the total number of deaths D (I) among the immigrants can be written in terms of an integral equation as

$$D(I) = \int_A^\alpha P[I(x)]\, \mu(x)\, dx \tag{36}$$

where $\mu(x)$ is the force of mortality. Since A is the lowest age of the immigrants and over time they age and reach the end of the life span, the limits of the integral age (A, α). The explicit forms of $P[I(x)]$ are given in (6) and (7) substitutions of which in (36) produces

$$D(I) = \int_A^B l(x)\, \mu(x) \int_0^{x-A} \frac{I(x-h)}{l(x-h)}\, dhdx + \int_B^\alpha l(x)\, \mu(x) \int_{x-B}^{x-A} \frac{I(x-h)}{l(x-h)}\, dhdx \tag{37}$$

Writing

$$\int_0^x \frac{I(x-h)}{l(x-h)}\, dh = P(x)$$

such that $$P'(x) = \frac{I(x)}{l(x)} \tag{38}$$

and remembering that by definition

$$l(x)\,\mu(x) = -\,l'(x) \tag{39}$$

we can express (37) as

$$D(I) = -\int_A^B P(x-A)\, dl(x) - \int_B^\alpha [P(x-A) - P(x-B)]\, dl(x)$$

$$= -\int_A^\alpha P(x-A)\, dl(x) + \int_B^\alpha P(x-B)\, dl(x) \tag{40}$$

which after integration by parts simplifies as

$$D(I) = -\Big[P(x-A)\, l(x)\Big]_A^\alpha + \int_A^\alpha l(x)\, \frac{I(x)}{l(x)}\, dx + \Big[P(x-B)\, l(x)\Big]_B^\alpha$$

$$-\int_B^\alpha l(x)\, \frac{I(x)}{l(x)}\, dx = \int_A^B I(x)\, dx = I \tag{41}$$

Thus the limiting value of the total number of deaths among the immigrants is exactly the same as the number of immigrants admitted over any time interval. Accordingly, the immigrant population P(I) can be regarded as a special kind of a stationary population in which the size reduced by deaths is continuously being replenished by the same number of immigrants.

In the example discussed in Section 2(a) where $I(x) = kl(x)$ for $A \leqslant x \leqslant B$ and $= 0$ otherwise, the expressions derived for I and $P(I)$ in (14) and (16) can be used to compute the crude death rate among the immigrants

$$d(I) = \frac{I}{P(I)} = \frac{T(a) - T(B)}{\int_A^B I(x)\, dx} \tag{42}$$

For the special case in which $A = 0$ and $B = \alpha$, the death rate can be expressed as a ratio of (17) and (19) when it simplifies into

$$d(I) = \frac{I}{\bar{x}} \tag{43}$$

where $\bar{x}$ is the average age of the stationary population. As is well known in demographic literature the death rate in a stationary population is given by $1/e(0)$ wheae $e(0)$ is the expectation of life at birth. When mortality is very high [$e(0) \leqslant 25$ years], $\bar{x}$ has generally been found to be greater than $e(0)$. Thereafter, the inequality gets reversed such that $e(0)$ and $\bar{x}$ both increase with decline in mortality in a manner that sets the limiting relationship between the two as $e(0) = 2\bar{x}$. Consequently, in low mortality examples, where the size of the immigrants entering at any time is proportional to the stationary population, the death rate among the immigrants is considerably greater than that suggested by the life table.

Expressions similar to (42) and (43) can be derived for the special cases discussed in Sections 2(b) and 2(c) and the same can be compared with those of the native population.

4. Births to Immigrants

Like deaths, a general expression for the total number of births to the immigrants can be written as

$$B(I) = \int_A^{\alpha} P[I(x)]\, m(x)\, dx \tag{44}$$

where A is the lowest age of the immigrants, α is the end of the life span and $m(x)$ is the age-specific fertility rate at age x.

For the particular case $A = 0$ and $I(x) = kl(x)$, $P[I(x)]\ kxl(x)$ (see eqn. 18) in which case we can write (44) as

$$B(I) = k \int_0^{\alpha} xl(x)\, m(x)\, dx \tag{45}$$

An alternative expression of (45) is

$$B(I) = kRT \tag{46}$$

where T is the average age of mothers and

$$R = \int_0^{\alpha} l(x)\, m(x)\, dx \tag{47}$$

is the net reproduction rate. Substituting (19) for P(I), the birth rate can be expressed simply as

$$b(I) = \frac{B(I)}{P(I)} = \frac{RT}{e(0)\,\bar{x}} \tag{48}$$

5. Births to Native Population Experiencing Below Replacement Fertility

It has been shown earlier (Espenshade *et* a/., 1982; Mitra, 1983) that when a constant number of immigrants *I* with a fixed age composition enter a country every year where the net reproduction rate *R(N)* is less than one, the limiting value of the annual number of births *B* is given by

$$B = \frac{B(I)}{1 - R(N)} \tag{49}$$

This formula holds even when the second and higher generation immigrants (but not the first) adopt the mortality and fertility patterns of the host country which do not change over time. That is to say, the immigrants themselves may have their own set of unchanging vital rates without any restriction on the numerical value of *R* and (49) will still be true with an appropriate value of *B(I).* If a generation higher than the second becomes the first to adopt the host country's vital rates, the annual number of births will still approach a limiting value that will have an expression different from (49).

Next, we obtain the limiting value of the number of births to native population from (49) as

$$B(N) = B - B(I) = \frac{B(I)\,R(N)}{1 - R(N)} \tag{50}$$

and the limiting size of the native born population (which is stationary) as

$$P(N) = Be(0, N) = \frac{B(I)\,e(0, N)}{I - R(N)} \tag{51}$$

where *e(0, N)* is the life expectancy of the native population. Therefore, the birth rate *b(N)* of the native population can be expressed as

$$b(N) = \frac{B(N)}{P(N)} = \frac{R(N)}{e(0, N)} \tag{52}$$

which can be compared with *b(I).* When *b(l)* is given by (48) and when the immigrants themselves adopt the host country's vital rates such that $R = R(N)$ and $e(0) = e(0, N)$ we get

$$\frac{b(I)}{b(N)} = \frac{T}{\bar{x}} \tag{53}$$

In most examples, average age of motherhood is less than the average age of the population. Therefore, in this special case, the birth rate of the immigrant population will be smaller than that of their native counterpart.

The overall birth rate b and its relationship between the same for the immi grants and the native population is quite interesting. This may be seen by first writing

$$b = \frac{B}{P(N) + P(I)} \tag{54}$$

and then as

$$b = \frac{B}{Be(0, N) + P(I)} \tag{55}$$

from (51). Rearranging terms, we can write (55) as

$$\begin{aligned} \frac{1}{b} &= e(0, N) + \frac{P(I)}{B} \\ &= e(0, N) + \frac{(1 - R(N))\, P(I)}{B(I)} \end{aligned} \tag{56}$$

from (49) and then as

$$\frac{1}{b} = \frac{R(N)}{b(N)} + \frac{1 - R(N)}{b(I)} \tag{57}$$

due to (52) and (48). That is to say, the overall birth rate b is the weighted harmonic mean of $b(N)$ and $b(I)$ with weights $R(N)$ and $1 - R(N)$ respectively. Observe that this relationship does not need any assumption about the age composition or the vital rates of the immigrant population.

Since the effect of the differential vital rates between the immigrant and the native population has been outlined before, we now choose to drop the distinction between $R(N)$ and R as well as that between $e(0, N)$ and $e(0)$ for operational simplicity and write only $e(G)$ and R whenever we have to refer to those parameters.

It is easy to see that α years must elapse after the start of the process (described by constant number of immigrants with fixed age composition and unchanging vital rates) for the immigrants to attain the ultimate size *P(I)*. Similarly, the annual number of births to the immigrants will attain its limiting value of *B(l)* after *v* years where *v* is the upper boundary of the reproductive age-interval. After another a years the second generation immigrants will reach itsulti mate size given by *B(l)* e(0). In the same way, the limiting size of the native born population including the second and succeeding generation of immigrants may be written as *Be(Q)*. Their ratio *B(I)/B* or 1 — *R* (see equation 49) determines the proportion that the second generation immigrants will bear to the total native population. Similar expressions can be derived for the third and subsequent generation of immigrants.

6. Deaths among the Native Population

As the number of births in the host country reaches the limiting value of *B,* the age composition reaches the limiting pattern given by *Bl(x)*. Therefore, the number of deaths *D(N)* among the native population also equals *B*. Note from (50) that of the *B* births, the number attributable to the native population is given by *B* — *B(l),* Thus, the excess of deaths over births which is *B(I)* in the native population *P(N)* is replaced by the same number of births to the immigrants and that is how the stationarity of *P(N)* is maintained. Simultaneously, as we noted earlier, the number of deaths *D(I)* among the immigrant population *P(I)* matches *I*, the number immigrating over any given time interval, making *P(I)* stationary in the process.

7. Time to Reach the Stationary State

At *ihh* point, a few words about the time intervals necessary for the different components of the population to attain their respective limiting values may be appropriate. As noted earlier, *P(l)* or the size of the immigrant population will stabilize after α or about 100 years and *B(l)*, the number of births to the immigrants after about 50 years from the start of the process. However, the birth trajectory will usually take much longer to stabilize. This may be seen by looking at an alternative picture of the same situation in which the host population is kept isolated from the start of the process. In that case, the continuation of a net reproduction rate of less than one will ensure extinction of that closed population in the long run.

This process leading towards extinction becomes apparent from the standard equation of the birth trajectory for the closed population which is given by

$$B(t) = Qe^{rt} + \ldots \qquad (58)$$

where r is the real root corresponding to the net reproduction rate R. Note that in our example r is always negative. As is well known, other terms of the series contain the exponential functions of the imaginary roots of the standard Lotka integral equation, the real parts of which get increasingly smaller and smaller than the real root r.

According to the mathematics of stable population the limiting value zero of the birth trajectory is reached at $t = \infty$ However, a realistic and finite measure of t can be obtained by solving for t corresponding to a preassigned small value of e^{rt} like say .05 which for r = —.01 turns out to be 300 years. At that point, contribution of the higher order terms of (58) will be small and can be neglected for all practical purposes

In order to study the trajectory of births to the immigrants and their descendants we begin by decomposing the births in terms of the order of generation to which they belong. After a certain length of time since the start of the process, the second generation of births, as we have noted earlier will be $B(I)$ in number. The same for the third generation will be $B(l)R$, for the fourth, $B(I)R^2$ and so on. The sum total of these components will reach the limiting value of

$$B = B(I)\,(1 + R + R^2 + \ldots) = \frac{B(I)}{1 - R}\,,\; R < 1 \qquad (59)$$

when t $\rightarrow \infty$ which is the same as (49) as it should be. Then the proportion that the second through the nth generation of births will bear to that limiting size can be obtained as 1 — R^{n-1}. To give a simple illustration let r be equal to —.01, the same value that we used earlier to trace the birth trajectory of the native population and further, let us assume that the length of a generation T is 25 years. This combination of r and T substituted in the standard formula

$$e^{rt} = R \qquad (60)$$

produces a value of .7788 for R, Then the number of generations that will account for 95 percent of the limiting value B can be solved from the equation

$$1\text{-}R^{n-1} = .95$$

which gives a value of 13 for n. Since the immigrants are assigned generation number one, the average interval between generations one and thirteen may be estimated as 12 T or about 30 years which is about the same we obtained earlier as the estimated time for virtual extinction of the native population. It may be noted that the mathematical relationship between r, T and R given by (60) assures such identity for all possible combinations of values of those parameters.

8. Summary and Concluding Remarks

The impact of continued immigration becomes increasingly significant under conditions of below replacement fertility i0 the host country. The impact is multidimensional although most of these dimensions appear to be highly correlated with one, namely, the demographic dimension. For purposes of demonstration we have chosen to look into the demographic impact of immigration under certain simplifying conditions like the constancy of the vital rates prevailing in the host country and the constancy of the number of immigrants at every age. The limiting size and age composition of the immigrant population can be derived years after of the process where a is the span of life. It has been shown that, thereafter, the number dying among the immigrants is the same as the number immigrating over any time interval Under below replacement fertility condition the difference between the number of deaths and births among the native population (which is positive) approaches its limiting value given by the total number of births to the immigrants. As a result, the population becomes stationary. Interestingly enough, the limiting value of the proportion of the population who are children of immigrants has been found $1 — R$ where R is the net reproduction rate. That is to say if $R = .8$, the second generation immigrants will account for 20 percent of the native population in the long run. These and other results assuming several alternative age compositions of the immigrants have been presented in the paper.

The operational procedure for determining the time required to reach stability is usually based upon a comparison of the real part of the largest complex root to the real root of r, the intrinsic rate of growth. In this example, the trajectory's approach to stationarity has been found to depend primarily on the contribution of the real root alone which is negative. The time for convergence is normally not small for the stable models and this one is no exception to that general rule. We have shown that for $r = — .01$ it could take as much as 300 years for the trajectory to come reasonably close to its limiting constant value.

Observe that in the derivation of most of our results we have applied the vital rates of the native population on the immigrants as well. If the immigrants are assumed to experience rates different from that of the host country, the numerical value of $B(I)$ will change without affecting the algebraic formulas of the relevant parameters derived in the paper as long as their children can be assumed to be subjected to the rates of the native population. If the adoption of the host country's rate has to be delayed any further, the formulas will undergo appropriate modifications but the basic findings concerning the eventual stationarity etc., will remain unaltered.

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